Mathematics 222B Lecture 3 Notes

Daniel Raban

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1 Approximation in Bounded Domains and the Extension Theorem

Today, our goals are

- Prove approximation (or density) theorems for Sobolev spaces.
- Prove extension theorems and the trace theorem (tools for dealing with $W^{k,p}(U)$ when U is a bounded domain).

1.1 Approximation theorems in bounded domains

Given $u \in W^{k,p}(U)$, we want to approximate it by something that is "better" (e.g. u is smooth or has a nice support property). Last time, we discussed two tools:

1. Convolution and mollification: If $f, g : \mathbb{R}^d \to \mathbb{R}$, then

$$f * g(x) = \int f(x-y)g(y)\,dy.$$

This has the property that

$$\partial_{x_j}(f * g)(x) = \partial_{x_j} f * g(x) = f * \partial_{\partial_{x_j}} g(x).$$

This means that you only need one of the functions to be smooth to get a smooth result.

For $\varphi \in C_c^\infty(\mathbb{R}^d)$, if we denote $\varphi_\varepsilon = \frac{1}{\varepsilon^d}\varphi(\cdot/\varepsilon)$, then

$$\varphi_\varepsilon f \xrightarrow{\varepsilon \to 0} f,$$

where the left hand side is smooth. If $f \in \mathcal{D}'(\mathbb{R}^d)$, this convergence is convergence of distributions, and if $f \in L^p(\mathbb{R}^d)$, this convergence is in L^p.

2. Smooth partition of unity: If $\{U_\alpha\}_\alpha \in A$ is a collection of open sets (usually $U \subseteq U_{\alpha\in A} U_\alpha$) then there exist functions $\chi_\alpha(x)$ ($\alpha \in A$) such that

   (i) χ_α is smooth.

   (ii) $\sum_{\alpha\in A} \chi_\alpha = 1$ on U, where for all $x \in U$, $\chi_\alpha(x) = 0$ except for finitely many α.

   (iii) $\operatorname{supp} \chi_\alpha \subseteq U_\alpha$.

Theorem 1.1. *Let $k \geq 0$ be an integer and $1 \leq p < \infty$.*

(i) $C^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.

(ii) $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.

Proof.

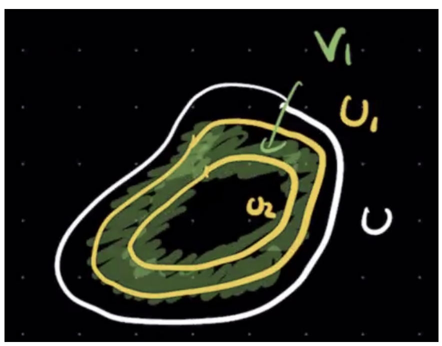
(a) This is an application of mollification

(b) Approximate by $f\chi(1/R)$, letting $R \to \infty$, where $\chi \in C_c^\infty(\mathbb{R}^d)$ is such that $\chi(0) = 1$. □

Theorem 1.2. *Let $k \geq 0$ be an integer, $1 \leq p < \infty$, and U an open subset of $\mathbb{R}^d$. Then $C^\infty(U)$ is dense in $W^{k,p}(U)$.*

Proof. Let $u \in W^{k,p}(U)$, and fix $\varepsilon > 0$. We want to find $v \in C^\infty(U)$ such that $\|u-v\|_{W^{k,p}} \leq \varepsilon$.

Define $U_j = \{x \in U : \operatorname{dist}(x, \partial U) > 1/j\}$, and let $V_j = U_j \setminus \overline{U_{j+1}}$



Then $U \subseteq \bigcup_{j=1}^\infty V_j$, so there is a smooth partition of unity χ_j subordinate to V_j. Now split

$$u = \sum_{j=1}^\infty \underbrace{u\chi_j}_{:=u_j}.$$

Then, as $\operatorname{supp} \chi_j \subseteq V_j$, we have that $\operatorname{supp} u_j = \operatorname{supp}(u\chi_j) \subseteq V_j$. Moreover, $u_j \in C_c^\infty(\mathbb{R}^d)$.

If we let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\int \varphi = 1$ and $\operatorname{supp}\varphi \subseteq B_1(0)$ is a mollifier, let $v_j = \varphi_{\varepsilon_j} * u_j$, where ε_j is chosen to achieve

$$\|u_j - v_j\|_{W^{k,p}} \le 2^{-j\varepsilon}, \qquad \operatorname{supp} v_j \subseteq \widetilde{V}_j = U_{j-1} \setminus \overline{U_{j+2}}.$$

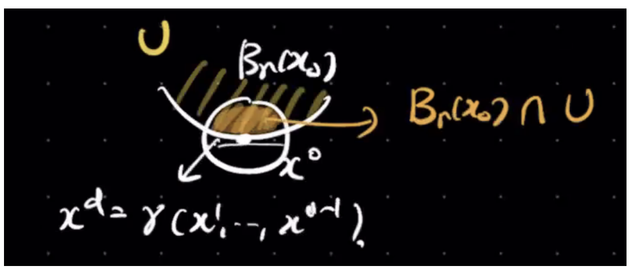
Here, we make use of the fact that $\operatorname{supp} f * g \subseteq \operatorname{supp} f + \operatorname{supp} g = \{x + y \in \mathbb{R}^d : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}$. Now take $v = \sum_{j=1}^\infty v_j$. This is well-defined, as $\widetilde{V}_j$ is locally finite. This is also smooth, so $v \in C^\infty(U)$. On the other hand,

$$\|v - u\|_{W^{k,p}} \sum_{j=1}^\infty \|v_j - u_j\|_{W^{k,p}} \le \sum_{k=1}^\infty 2^{-j}\varepsilon = \varepsilon. \qquad \square$$

Theorem 1.3. *Let $k \ge 0$ be an integer, $1 \le p < \infty$, and U a bounded open set with ∂U of class C^1. Then $C^\infty(\overline{U})$ is dense in $W^{k,p}$.*

Here, $C^\infty(\overline{U})$ is the set of functions $u : U \to \mathbb{R}$ such that u is the restriction to U of a smooth function $\widetilde{u} \in C^\infty(\widetilde{U})$, where $\widetilde{U} \supseteq \overline{U}$ is open.

Definition 1.1. We say that ∂U is **of class** C^k if for all $x_0 \in \partial U$, there exists a radius $r = r(x_0) > 0$ such that, up to relabeling the variables, $B_r(x_0) \cap U = \{x \in B_r(x_0) : x^d > \gamma(x^1, \ldots, x^{d-1})\}$ for some C^k function $\gamma = \gamma(x^1, \ldots, x^{d-1})$ on $B_r(x_0) \cap (\mathbb{R}^{d-1} \times \{x_0^d\})$.



For the proof, we want to apply mollification, but the difficulty is what happens near the boundary. The idea is to first look at a small piece of the boundary at a time.

Proof. Step 1: Let $u \in W^{k,p}(U)$. By the definition of C^1-regularity of ∂U, ∂U can be covered by balls $\{B_{r_k}(x_k)\}_{k=1}^K$, in each of which U can be represented as the region above some C^1 graph. The number of such balls, K, is finite by the compactness of ∂U. We may add to $U_k = B_{r_k}(x_k)$ an open set U_0 which contains $U \setminus \bigcup_{k=1}^K U_k$, so that $\{U_0, U_1, \ldots, U_k\}$ is an open covering of U.

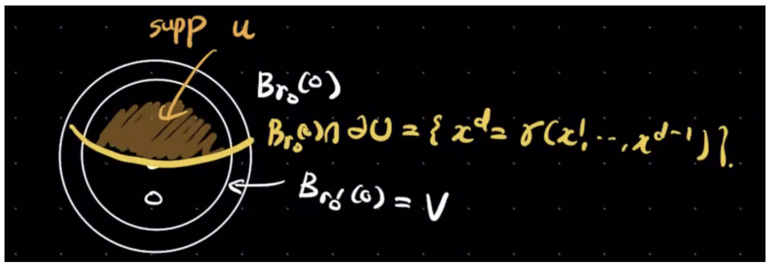
Let $\{\chi_k\}_{k=0}^K$ be a smooth partition of unity subordinate to $\{K_k\}_{k=0}^K$, and split

$$u = \sum_{k=0}^{\infty} u\chi_k =: u_0 + \sum_{k=1}^{K} u_k.$$

Here, u_0 is compactly supported, and $u \in W^{k,p}(\mathbb{R}^d)$, so we can use mollification, as before.

To deal with the u_k with $k \geq 1$, it suffices to consider the case where $U = B_{r_0}(x_0)$ and $\operatorname{supp} u_k \subseteq V \subseteq U$, where V is a smaller ball $B_{r_0'}(x_0)$, in which $B_{r_0}(x_0) \cap \partial U$ is more concrete.

Step 2: Without loss of generality, assume $x_0 = 0$.



We use a two-step approximation. Let $\varepsilon > 0$.

1. Let $w_\eta(x) = u(x + \eta e_d)$, where $e_d = (0, 0, \ldots, 0, 1)$, and η will be chosen. Then $\operatorname{supp} w_\eta$ is the support of u shifted by 1. For η small enough, we have

$$\|u - w_\eta\|_{W^{k,p}(U \cap B_{r_0}(0))} < \frac{1}{2}\varepsilon.$$

   Moreover, ε is defined on $B_{r_0}(0) \cap U - \eta e_d$

2. Let $v = \varphi_\delta * w_\eta$, and if $\delta \ll \eta$ (and $\operatorname{supp} \varphi \subseteq B_1(0)$), then v is well-defined on $V \cap \{x^d > \gamma(x^1, \ldots, x^{d-1})\}$. And if δ is sufficiently small, then

$$\|v - w_\eta\|_{W^{k,p}(U \cap B_{r_0}(x_0))} < \frac{1}{2}\varepsilon.$$

   This gives us

$$\|u - v\|_{W^{k,p}(U)} \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

   Moreover, $v \in C^\infty(\overline{V \cap \{x^d > \gamma(x^1, \ldots, x^{d-1})\}})$, which is acceptable. $\square$

1.2 The extension theorem

The extension theorem is a tool to deal with $u \in W^{k,p}(U)$, where U is a bounded domain, by producing an extension of $u \in W^{k,p}(\mathbb{R}^d)$ with quantitative bounds on the extension.

Theorem 1.4 (Extension theorem)**.** *Let $k \geq 0$ be a nonnegative integer, $1 \leq p < \infty$, U a bounded domain with with C^k boundary. Let V be an open set such that $V \supseteq \overline{U}$. Then there exists an operator $\mathcal{E} : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$ such that*

(i) (Extension) $\mathcal{E}u|_U = u$.

(ii) (Linear and bounded) $\mathcal{E}$ is linear, and $\|\mathcal{E}u\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(U)}$.

(iii) (Support prescription) $\operatorname{supp} \mathcal{E}u \subseteq V$.

Proof. Observe that, by the previous approximation theorem, it suffices to consider $u \in C^\infty(\overline{U})$ (by density and the boundedness property (ii)).

Step 1: (Reduction to the half-ball case) As in Step 1 in the proof of the previous theorem, construct the open sets $U_0, U_1, \dots, U_K$ and the partition of unity $\chi_0, \chi_1, \dots, \chi_k$. Define $u_k = \chi_k u$, and observe that

- u_0 is already in $W^{k,p}(\mathbb{R}^d)$ and $\operatorname{supp} u_0 \subseteq U_0 \subseteq V$,
- $u_k \in C^\infty(\overline{U})$, and $\operatorname{supp} u_k \subseteq B_{r_0} \subseteq U_k \cap U$.

Observe that if we change variables

$$\begin{cases} y^j = x^j - x_0^j & \text{for } j = 1, \dots, d-1, \\ y^d = x^d - \gamma(x^1, \dots, x^{d-1}), \end{cases}$$

then $U_k \cap U$ gets mapped into $\{y \in B_{\widetilde{r}}(0) : y^d > 0\}$.

Note that the change of variables $x \mapsto y$ is C^k, and u_k is smooth, so $u_k(y) = u_k(x(y))$ satisfies, by the chain rule,

$$\|u_k(y)\|_{W^{k,p}_y(\widetilde{U})} \leq C\|u_k(x)\|_{W^{k,p}_x}.$$

Step 2: (Extension in the half-ball case) Now we have $U = B_r^+(0)$, $W = B_{r/2}^+(0)$, and $\operatorname{supp} u \subseteq W$, and we want to extend u. The idea is the *higher order reflection method.* Define

$$\widetilde{u} = \widetilde{u} = \begin{cases} u & x^d > 0 \\ \sum_{j=0}^K \alpha_j u(x^1, \ldots, x^{d-1}, -\beta_j x^d) & x^d < 0, \end{cases}$$

where the scaling factor $0 < \beta_j < 1$ is chosen so that $(x^1, \ldots, x^{d-1}, -\beta_j x^d) \in B_r^+(0)$. We need to match the normal derivatives on $\{x^d = 0\}$ up to order k. Observe that $\partial^j_{x^d}(u(x^1, \ldots, x^{d-1} - \beta_j x^d)) = (-1)^j \beta_j^j (\partial^j_{x^d} u)(x^1, \ldots, x^{d-1}, -\beta_j x^d)$. We get

$$\begin{cases} u(x^1, \ldots, x^{d-1}, 0+) = \sum_{j=0}^K \alpha_j u(x^1, \ldots, x^{d-1}, 0-), \\ \partial_{x^d} u(x^1, \ldots, x^{d-1}, 0+) = \sum_{j=0}^k \alpha_j (-\beta_j)(\partial_{x^d} u)(x^1, \ldots, x^{d-1}, 0+) \\ \vdots \\ \partial^k_{x^d} u(x^1, \ldots, x^{d-1}, 0+) = \sum_{j=0}^k \alpha_j (-\beta_j)^k (\partial^k_{x^d} u)(x^1, \ldots, x^{d-1}, 0+). \end{cases}$$

This is equivalent to

$$\begin{cases} 1 = \sum_{j=0}^K \alpha_j \\ 1 = \sum_{j=0}^K \alpha_j(-\beta_j) \\ \vdots \\ 1 = \sum_{j=0}^K \alpha_j(-\beta_j)^K. \end{cases}$$

Written in matrix form, this is a linear system involving a **Vandermonde matrix**

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ -\beta_0 & -\beta_2 & \cdots & -\beta_K \\ \vdots & \vdots & \vdots & \vdots \\ (-\beta_0)^K & (-\beta_2)^K & \cdots & (-\beta_K)^K \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_K \end{bmatrix}.$$

Now use that fact that if all the β_j are distinct, then this matrix is invertible. This means that there is a choice of $(\alpha_0, \ldots, \alpha_K)$ so that these equations hold. This defines $\widetilde{u}$ on $B_r(x)$ which extends u and matches all derivatives up to order K on the boundary $\{x^d = 0\}$. Finally, put an appropriate smooth cutoff $\chi_V = 1$ on U with $\operatorname{supp} \chi_V \subseteq V$ to define $\mathcal{E}u$, i.e. $\mathcal{E}u = \chi_V \widetilde{u}$. $\square$